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ON THE GROWTH AND RISE OF INDIVIDUAL VAPOUR BUBBLES IN NUCLEATE POOL BOILING

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Abstract—The following assumptions are made for the case of nucleate pool boiling in the regime of individual vapour bubbles: (a) rising bubbles carry with them thin boundary layers of superheated liquid, broken away at their departure, from the boundary layer of the heating surface; (b) the disturbances caused within the bulk by the bubbles are the same as if the bubbles were solid bodies of the same shape and size.

Using the theory of virtual masses and the above assumptions, relations for the growth rate and rise velocity of the bubbles are developed, which are found to be in satisfactory agreement with available experimental data.

NOMENCLATURE

A,	area of surface;
a,	thermal diffusivity;
a, b,	semi-axes of a planetary ellipsoid;
B, C, c,	numerical coefficients;
c_L,	specific heat of the liquid;
D_0,	average diameter of a departing bubble;
e,	eccentricity of the ellipsoid;
F,	buoyancy force of a bubble;
F',	adhesive force of a stalked bubble;
f,	frequency of bubble production in a centre;
g,	acceleration of gravity;
h,	surface heat-transfer coefficient;
h,	height of bubble centre over hot plate;
I_p,	Bessel function of order p and imaginary argument;
k,	coefficient of thermal conductivity;
L,	latent heat of vaporization;
l,	representative length;
M,	mass;
M_v,	virtual mass;
m, n, q,	numbers;
p_v, p_s,	vapour and saturation pressures respectively;
Q,	heat flux;
r,	radius;
R, R',	radii of interfaces of a bubble boundary layer;
$\dot{R}$,	growth rate of a bubble;
s,	average spacing between neighbouring active centres;
T,	kinetic energy;
t,	temperature;
u,	mean velocity within the boundary layer of the heating surface;
v,	rising velocity of a bubble;
W,	resistance due to friction;
w,	expansion velocity of a boundary layer;
x, y, z,	variables;
$\dot{x}$,	time derivative of quantity x.

Greek symbols

σ,	surface tension coefficient;
α, β, γ, κ,	numerical coefficients or exponents;
δ,	thickness of a boundary layer;
μ,	coefficient of apparent mass;
ρ,	density;
θ,	contact angle in sexagessimal degrees;
τ,	time;
ζ,	coefficient of friction;
ν,	coefficient of dyanmic viscosity.

Subscripts

b,	bulk conditions;
e,	expansion;
L,	liquid;
o,	departure conditions;
s,	saturation;

t,	translation;
tr,	transverse;
v,	vapour;
w,	wall conditions.

1. INTRODUCTION

1.1 *General remarks*

NUCLEATE pool boiling in the regime of isolated bubbles is characterized by the periodic formation of individual vapour bubbles in certain of the small cavities of the heating surface, where gas or vapour microbubbles, previously retained there, serve as nuclei.

These cavities, also called "nucleating or active centres" are distributed at random over the heating surface or plate and it is not yet possible to predict whether or not particular cavities of this surface will become active, or at what moment they will begin or cease to be active.

One could see the formation of bubbles suddenly ceasing in some of the active centres, without any apparent reason, the activity being immediately taken over by neighbouring centres or even, after a short delay, resumed by the initial ones.

It is now well established that bubbles do not form without the presence of nuclei (microbubbles of gas or vapour or very small solid particles in suspension) and also that bubbles successively leaving the same cavity are not equal nor are they equal to those leaving neighbouring cavities.

The density of active centre distribution over the heating surface, the frequency of bubble production at a given centre, the contact times of the bubbles and the magnitude and shape of the bubbles all depend on a multitude of more or less independent parameters such as the microgeometry of the wetted face of the heating plate, its condition and nature, the physical properties of the bulk liquid, the pressure and temperature fields, the heat flux intensity, the motion of the liquid in the vicinity of the heating plate. This simple enumeration is sufficient to emphasize the statistical character of nucleate pool boiling.

No wonder therefore, that in spite of the several papers dedicated in recent years to the study of this complex phenomenon, most of which are of remarkable value, progress is still slow. Attempts are being made to penetrate the intimate details of the phenomenon of boiling, to understand what is happening in the cavities, grooves or pores of the heating surface, to detect the mechanism of bubble formation, to know the manner in which gas or vapour microbubbles are entrapped in the cavities of the heating surface [1]. This kind of research will certainly not fail to elucidate certain facets of the phenomenon. However, complete understanding has not been reached even of certain macroscopic aspects of nucleate pool boiling, such as the growth and evolution of isolated vapour bubbles.

It is the aim of this paper to try to improve in some measure the knowledge of these particular aspects.

Recently it was suggested that a close analogy exists between natural convection and nucleate pool boiling in the region of isolated bubbles [2–4].

Based on this analogy, it was possible to set up a sufficiently close representation of the phenomenon of nucleate pool boiling in the region of isolated bubbles.

Research has shown that the bubble formation is a step-by-step process. During a certain preliminary time τ_d, the bubble still visually unperceivable, grows in its generating cavity pushed forward by the nucleus existing there. Then it appears at the surface of the plate, where, still attached, it continues to grow, now covered by the strongly superheated boundary layer of the heating surface, from which it gets most of the heat and all of the vapour it needs for growing.

While growing, the bubble impinges upwards on the covering layer, which results in a weak ascending current of superheated liquid above the active centre.

This second phase of the bubble evolution has a duration τ_b lasting until the bubble has reached a certain critical volume V_0. Then the buoyancy-force, together with the action of the surrounding liquid, overcomes the adhesive forces and the bubble detaches and starts its rising motion. But generally a little before its break-off the bubble develops a thin pedicular stem, by which it remains attached to the heating surface while its spherical part begins to rise. The pedicle, extended by the rise of the bubble, begins to shrink and at a certain moment breaks.

As more comprehensively shown below, the bubbles departing from an active cavity carry along with them certain amounts of superheated liquid, drawn from the boundary layer of the heating surface, Relatively cooler bulk liquid replaces them instantly, surrounding the quite small lower part of the broken pedicle, which then contracts and possibly condenses partly. Thus only a tiny amount of vapour remains inside the cavity, where it becomes the nucleus of the next bubble.

Meanwhile the upper part of the broken stem rapidly re-enters, like a recoiling spring, into the bubble, forming a concavity on its lower part. This concavity, together with the effect of the resistance of the bulk liquid—which arises as soon as the bubble begins to move—is at the origin of the known lenticular shape which the bubble reaches almost immediately after its break-off and which it keeps during almost its whole rise. Only when it approaches the free top surface of the liquid, the bubble tends to reach again the spherical shape, owing to the rapid decrease of the hydrostatic pressure there. Figure 1 shows the different stages of the evolution of a bubble, from the moment shortly before its break off up to the formation of the lenticular shape.

After breaking off, the bubble follows the path of the weak current of superheated liquid developed over its generating cavity. While rising it pushes forward above it, and drags laterally by friction and in its wake by suction, certain quantities of superheated liquid drawn, at its departure, from the boundary layer of the heating surface, by which it was covered before starting its rise.

In this way an ascending current of superheated liquid begins to circulate around and behind the rising bubble. Part of the liquid drawn by this current masses together against the bubble and forms the bubble's own boundary layer, which will feed it with heat and vapour during the rise. The remainder of the superheated liquid, rising in the trace of the bubble, diffuses progressively into the bulk, giving up its superheat. This amount of heat, together with the heat pentrating directly by convection from the heating plate into the bulk, is the origin of the slight superheat of 0·3 to 0·6 degC of the bulk liquid, first observed by Max Jakob and coworkers. The bubble and its associated currents are replaced by descending currents from the bulk. The relatively cooler liquid of these currents enters the boundary layer of the heating plate, and whilst renewing its content, ensures the continuity of the process (Fig. 2).

It seems feasible to assume that the thickness of the bubble's own boundary layer, at the beginning of its rise, is of the same order of magnitude as that of the layer which covered the bubble before its break-off.

During the rise of a bubble the boundary layer of superheated liquid surrounding it may grow or decay depending upon the local conditions in the bulk through which the bubble is

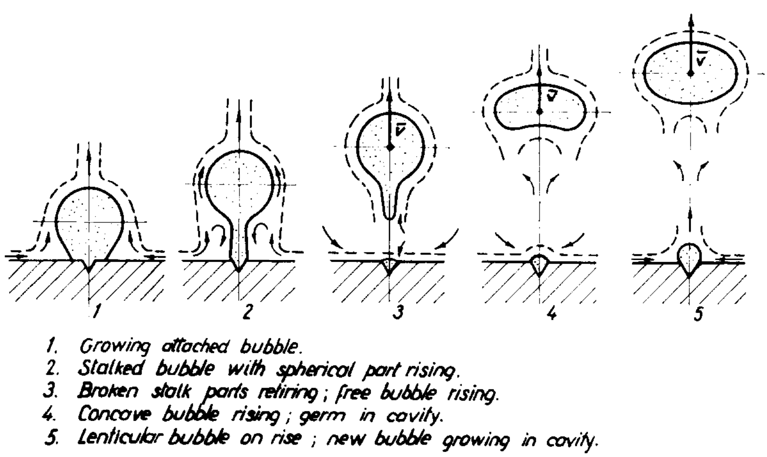

FIG. 1

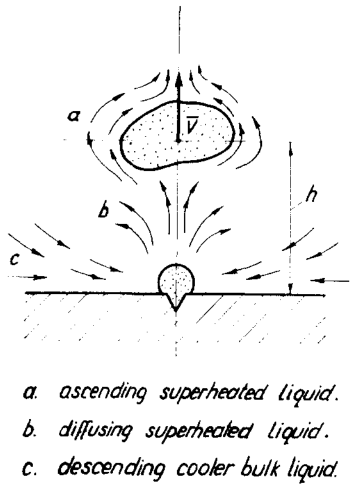

FIG. 2

passing and also upon conditions within the bubble boundary layer itself.

When a bubble enters into a superheated region of the bulk, it continues to grow, while rising, and bursts when reaching the free surface of the liquid. Alternatively, when it penetrates into a subcooled region it begins to reduce its growth rate and after reaching a certain maximum volume, as a result of the stored heat content of its boundary layer, it begins to shrink and finally collapses by condensation.

1.2. *Motion of vapour bubbles and virtual mass theory*

The behaviour of an isolated vapour bubble during its rising motion is hydrodynamically similar to that of an immersed solid body moving in a fluid medium. The large deformations of the bubbles during their motion are proof that, like solid bodies, they too are subjected to interactions with the bulk liquid. These interactions cannot differ from those which solid bodies of the same shape and magnitude undergo when moving with the same velocities across the bulk.

It seems suitable, therefore, to have recourse to the theory of virtual masses for vapour bubbles in exactly the same manner as for immersed solid bodies. However, it must be taken into account that, unlike solid bodies, the vapour bubbles change their size and shape during their motion.

It has long been known that an immersed body in *non-uniform motion within a fluid moves as if its own mass has been increased by an additional one*, called "the apparent or associated mass". This may be explained as follows: The movement of the immersed body produces disturbances which propagate like waves through the entire bulk liquid, regardless of its extent. For the acceleration of the liquid involved in these disturbances a certain amount of kinetic energy must be spent; the apparent mass, when moving with the velocity of the solid body, would require this amount of energy.

Let M be the mass of the solid body, V' the volume of its immersed part, M' the mass of the displaced liquid and M_a the apparent mass; then

$$M_a = \mu\, M' = \mu \rho_L\, V'. \qquad (1\text{-}1)$$

For the total fictitious mass, called "virtual mass", we can write

$$M_v = M + M_a = M + \mu \rho_L\, V'. \qquad (1\text{-}2)$$

This virtual mass is introduced into the equations of motion of the immersed body or the bubble.

The coefficient of proportionality μ which appears in (1-1) and (1-2), called "the coefficient of apparent masses", depends on (a) the shape and size of the immersed part of the solid body, and (b) the kind of motion of the body. Generally of a tensoral character, the coefficient μ reduces to a scalar if the motion is a simple translation without rotation or a radial expansion (dilatation) [5].

For such motions the definition of the coefficient μ is expressed by the relation

$$\mu = \frac{M_a}{M'} = \frac{\frac{1}{2} M_a\, v^2}{\frac{1}{2} M'\, v^2} = \frac{\text{kinetic energy of disturbances}}{\text{kinetic energy of displacement}} \qquad (1\text{-}3)$$

During the greater part of their rise through the bulk liquid, the bubbles maintain an almost lenticular shape. The closest regular geometric body to this shape is the oblate spheroid (planetary ellipsoid), for which we shall now determine the coefficient of apparent masses μ. Since the vapour bubbles are performing a compound motion composed of a translation and a simultaneous expansion, we shall determine first the coefficients of apparent masses for each of these two motions separately and then calculate a coefficient for the compound motion.

(*a*) *Translation.* Let a be the equatorial and b the polar semi-axis of a planetary ellipsoid, moving along its polar axis. The coefficient of apparent masses for a translation μ_t, is [6]

$$\mu_t = \frac{a}{b} \cdot \frac{e - (\arc\sin e)(1 - e^2)^{\frac{1}{2}}}{(\arc\sin e) - e(1 - e^2)^{\frac{1}{2}}} = \frac{a}{b} \cdot \frac{e - (\arc\sin e)(b/a)}{(\arc\sin e) - e(b/a)}$$

where $e = (1 - b^2/a^2)$ is the eccentricity, $b \leqslant a$.

Of special interest for future derivations is the case where

$$a/b \approx 2{\cdot}4$$

which corresponds to the average lenticular bubble. For this particular case it yields

$$\mu_t = 1{\cdot}34. \tag{1-4}$$

(*b*) *Expansion.* Departing from the definition given above, the coefficient of apparent masses μ_e, for an expanding, spherically shaped body, will be

$$\mu_e = M_e/M'$$

where M_e is the apparent mass corresponding to a radial expansion of the body and M' the displacement of the body.

The kinetic energy of the disturbed liquid is

$$T_e = \tfrac{1}{2} \int_R^\infty \dot{r}^2 \rho_L \, dV = 2\pi \rho_L \int_R^\infty \dot{r}^2 \times r^2 \, dr$$

where

R is the radius of the sphere;

$dV = 4\pi r^2 \, dr$, the volume of an element of disturbed liquid;

$\dot{r}$ is the radial velocity of the element.

Since the equation of continuity is

$$4\pi r^2 \times \dot{r} = 4\pi R^2 \times \dot{R},$$

it follows that

$$T_e = 2\pi \rho_L \int_R^\infty (\dot{r}^2 r^4) \, dr/r^2 = 2\pi \rho_L R^4 \times \dot{R}^2 \int_R^\infty dr/r^2 = 2\pi \rho_L R^3 \times \dot{R}^2 = \tfrac{3}{2} M' \dot{R}^2$$

On the other hand, according to the definition of the apparent masses

$$T_e = \tfrac{1}{2} M_e w^2$$

where w is the velocity of expansion of the sphere, i.e.

$$w \approx \dot{R}.$$

Subsequently

$$T_e = \tfrac{1}{2} M_e \dot{R}^2 = \tfrac{3}{2} M' \dot{R}^2$$

so that

$$M_e = 3M'$$

and therefore

$$\mu_e = M_e/M' = 3. \tag{1-5}$$

(*c*) *Resultant motion.* The bubbles perform a motion consisting of an almost rectilinear translation combined with a simultaneous radial expansion, both without rotation. Generally the translation is the dominant motion.

Define a coefficient of apparent mass for the compound motion as

$$\bar{\mu} = \frac{\text{total kinetic energy of the disturbances}}{\text{total kinetic energy of the displacements}} = \frac{\frac{1}{2} M_t v^2 + \frac{1}{2} M_e w^2}{\frac{1}{2} M' v^2 + \frac{1}{2} M' w^2}$$

If the translation is the dominant motion, then $w^2 \ll v^2$ and

$$\bar{\mu} = \frac{M_e}{M'} + \frac{M_e w^2}{M' v^2} = \mu_t + \frac{w^2}{v^2} \cdot \mu_e \tag{1-6}$$

For boiling water the velocity of translation in the vicinity of the plate is of the order of $v = 0{\cdot}18$ m/s whereas the velocity of expansion

$$w \approx \dot{R} \approx 0{\cdot}04\text{–}0{\cdot}05 \text{ m/s}$$

so that an average value for the ratio w/v is

$$w/v \approx 1/4$$

and one may assume that the same ratio remains valid for other vapours too (See Section 5).

For the lenticular bubbles a mean value of a/b is 2·4, thus, as previously shown, $\mu_t = 1{\cdot}34$.

If it is assumed that for such shapes a value of 3 for μ_e may be maintained, then the value of the coefficient of apparent mass for the compound motion, in the vicinity of the plate, is

$$\bar{\mu} = 1{\cdot}53$$

or rounding off

$$\bar{\mu} = 1{\cdot}50 \tag{1-7}$$

a value which will be adopted in later calculations.

For increasing time, v increases whereas $\dot{R}$ decreases owing to the growth of the bubble volume, thus the second term on the right-hand side of (1-6) decreases rapidly so that

$$\lim_{\tau \to \infty} \bar{\mu} = \mu_t. \qquad (1\text{-}8)$$

1.3. *Thickness of the boundary layers*

The superheated layer circulating over the heating surface is in some measure similar to the boundary layer developed over a flat plate by a laminar steady state parallel flow. Therefore its thickness will be approximately [7]

$$\delta = 5{\cdot}8\, l\, Re^{-\frac{1}{2}}, \quad Re = l\bar{u}/\nu$$

where

δ is the thickness of the boundary layer;
l is the characteristic length;
$\bar{u}$ is the mean velocity within the layer;
ν is the dynamic coefficient of viscosity;

It was shown by Zuber [2], Hsu and Graham [8], Gaertner [9] and Kutateladze [10] that in order to maintain the production of individual bubbles in nucleate pool boiling, it is necessary that the spacing between neighbouring active centres does not become lower than 1·5–2·0 D_0, where $D_0 = 2R_0$ is the diameter of an average departing bubble (Fig. 3).

For water 100°C and 1 atm pressure with a mean spacing of $s = 3{\cdot}5\ R_0$ the characteristic length appearing in the Reynolds number becomes $l = \frac{1}{2}s - R_0 = 0{\cdot}75\ R_0$ so that with a mean velocity of $\bar{u} = 5$ cm/s and a coefficient $\nu = 3{\cdot}103$ cm²/s it follows that

$$\delta/R_0 = 1/2{\cdot}86. \qquad (1\text{-}9)$$

Gunther and Kreith [11] and Forster [12] arrive by different ways at the value

$$\delta/R_0 = 1/3$$

which is of the same order of magnitude.

As for the assumed mean value of $\bar{u} = 5$ cm/s, it may be observed that it is of the same order of magnitude as the growth-rate of the attached bubble. Indeed assuming that the bubble, while growing, displaces the liquid of the thin superheated layer which sweeps the plate, the following is obtained:

$$2\pi\, r\, \delta\, \bar{u} = \pi\, R_0^2\, \dot{R}_0$$

and with

$$r \approx 1{\cdot}5\ R_0 \quad \text{and} \quad \delta \approx R_0/3$$

it yields

$$\bar{u} \approx \dot{R}_0$$

and as shown later, $\dot{R}_0$ has a value of almost 5 cm/s for attached bubbles.

Consider now the boundary layer of a rising bubble. Up to its break-off, the bubble is covered by the boundary layer of the heating surface. When it departs, it is from this layer that the bubble acquires the amount of liquid from which it will make its own boundary layer. It seems, therefore, justified to assume that at least at departure the thickness δ' of the boundary layer of a bubble will be almost the same as that of the boundary layer of the heating surface, i.e.

$$\delta' \approx \delta. \qquad (1\text{-}10)$$

Turning now to the virtual mass concept, it ensues that the imaginary layer obtained when applying the apparent mass uniformly over the bubble interface, would have the same order of magnitude as (1-9) or (1-10). Indeed if δ'' is the thickness, R the internal radius and R' the external radius of this imaginary layer, then

$$\delta'' = R' - R = [(1 + \bar{\mu})^{1/3} - 1]\, R = \kappa\, R$$

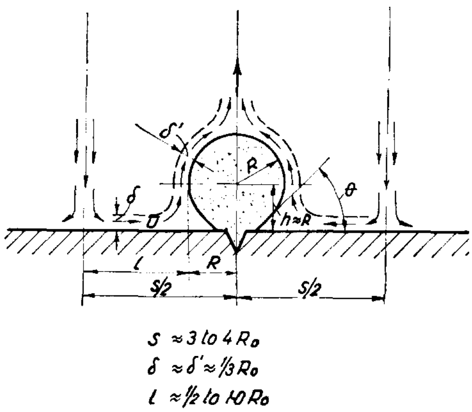

FIG. 3

and with $\bar{\mu} = 1{\cdot}5$ in the vicinity of the plate, it results that

$$\delta'' = R/2{\cdot}81 \tag{1-11}$$

i.e. almost the same value as above.

This close agreement is, however, only coincidental, but the order of magnitude of the different thicknesses must be the same. This may be explained as follows:

The concept of a dynamic boundary layer, and subsequently that of a thermal one, was introduced with the aim of avoiding the considerable difficulties encountered in the integration of the original Navier–Stokes equations for turbulent flows. But even the simplified equations of Prandtl cannot generally be integrated without having recourse to certain empiric laws for the distribution of the velocities within the boundary layer. Since for this distribution numerous different laws have been proposed, a wide range of different values for the same thickness of the boundary-layer have been obtained.

However, regardless of the method used to determine the velocity distribution, the corresponding boundary layer thicknesses are of the same order of magnitude.

On the other hand, the concept of perfect fluids, i.e. fluids deprived of any internal friction, has been proved, for example, to be a very effective instrument in most of the problems regarding navigation, since the viscosity of the water is very low.

But special problems, e.g. those of the motion of immersed bodies, cannot be solved without, directly or indirectly, considering the viscosity effects. It was therefore necessary to devise a conceptual framework that would take the viscosity effects into account, without discarding so convenient an assumption as that of perfect fluids.

As a result, the virtual mass concept was developed. The value of the imaginary masses introduced by virtue of this concept is largely dependant upon the shape, size and motion of the immersed body, all of which are only very approximately known in the case of vapour or gas bubbles.

Nevertheless, if related to the same motion, both approaches must lead to results, not identical, evidently, since means and methods are different, but of the same order of magnitude, because they are both taking into account the same viscosity effects.

It is therefore permissible to assume that between the volume of the bubble and of its boundary layer there must exist the relation

$$V_{\text{layer}} \approx \mu V_{\text{bubble}}$$

with μ having the same value as that for the corresponding apparent mass, or one very close to it.

2. MOTION AROUND THE CENTRE OF GRAVITY OF A BUBBLE

Consider an expanding spherical vapour bubble (Fig. 4) and let R and R' be the internal

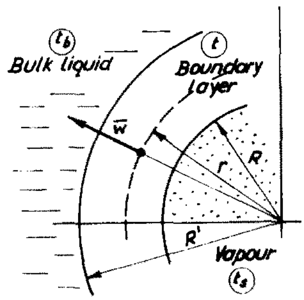

FIG. 4

and external radii of an imaginary layer replacing the apparent mass of the bubble.

Neglecting the mass of the vapour itself and omitting the translation of the bubble (since only the motion around the centre of gravity is being considered) as well as the heat and mass transfer due to the vaporization taking place at the vapour–liquid interface, the following energy equation may be written:

variation of the kinetic energy of the apparent mass = work done by pressure less work done by surface tension.

With

$V = \frac{3}{4}\,\pi\,R^3$, the volume of the bubble;
$A = 4\,\pi\,R^2$, the area of the interface of the bubble;
w, the radial velocity of a particle of the imaginary layer;
p_v, p_s, the vapour and saturation pressures;
σ, the surface tension coefficient;

the above equation becomes

$$d(\mu_e\, \rho_L\, .\, V\, .\, w^2/2) = (p_v - p_s)\, dV - \sigma\, dA. \tag{2-1}$$

Owing to the insignificant thickness of the imaginary layer

$$\delta = (R' - R) \approx R/3$$

we may assume that the radial velocity of a particle of this imaginary layer is approximately equal to that of the points of the vapour-liquid interface, i.e.

$$w \approx \dot{R} \tag{2-2}$$

so that equation (2-1) becomes

$$\tfrac{1}{3}\, \mu_e(\ddot{R}R + \tfrac{3}{2}\, \dot{R}^2) = (p_v - p_s)/\rho_L - 2\sigma/\rho_L\, R. \tag{2-3}$$

Introducing in this equation the value (1-5) of the coefficient of apparent masses for expansions, i.e.

$$\mu_e = 3$$

we obtain the equation

$$R\ddot{R} + \tfrac{3}{2}\, \dot{R}^2 + 2\sigma/\rho_L\, R = (p_v - p_s)/\rho_L \tag{2-4}$$

which is the well-known equation of Rayleigh [13].

As to the relative importance of the different terms of this equation, it may be pointed out that the effect of the surface tension may always be neglected in comparison with that of the pressure difference. This statement can be justified by the following considerations.

The so-called "critical radius" of a gas bubble, i.e. the radius of a bubble at the moment it starts to grow, as given by the formula of Laplace–Kelvin is

$$R_{\min} = \frac{2\sigma}{p_v - p_s}\, .\, \frac{\rho_L}{\rho_L - \rho_v} \approx 2\sigma/(p_v - p_s)$$

since $\rho_v \ll \rho_L$.

Subsequently

$$2\sigma/\rho_L\, R \approx \frac{p_v - p_s}{\rho_L}\, .\, \frac{R_{\min}}{R} \ll (p_v - p_s)/\rho_L$$

since it is well-known that

$$R_{\min} \ll R.$$

The last term on the left-hand side of (2-4) may therefore be omitted so that the equation becomes

$$R\ddot{R} + \tfrac{3}{2}\, \dot{R}^2 \approx (p_v - p_s)/\rho_L.$$

3. GROWTH OF VAPOUR BUBBLES

Neglecting the influence of buoyancy and friction, one may write the following energy equation:

heat passing across the interface plus kinetic energy of bubble expansion = heat of vaporization entering the bubble plus work of pressure less work of surface tension.

Explicitly written, the equation is

$$kA \left(\frac{\partial t}{\partial r}\right)_{r=R} \times d\tau + d(\mu_e\, V\rho_L \times w^2/2) = L\, \rho_v\, dV + (p_v - p_s)\, dV - \sigma\, dA \tag{3-1}$$

Making use of equation (2-1) the above expression reduces to the purely thermal equation

$$k\, A \left(\frac{\partial t}{\partial r}\right)_{r=R} \times d\tau = L\, \rho_v\, dV \tag{3-2}$$

In order to integrate this apparently simple equation, one must have recourse to the equation of Navier–Stokes, the equation of continuity as well as to the heat conduction equation of Fourier–Kirchhoff. Apart from the complexity of the calculus, the various simplifications which must be introduced will lead finally to an approximate solution. It seems therefore adequate to start from the beginning with a reasonable approximation of the temperature field within the boundary layers of the bubbles.

Since the works of Bošnjaković [14] and M. Jakob [15], several studies have been devoted to the pursuance of a relation for the growth of a bubble during its rise. Having gathered a large quantity of experimental material, W. Fritz [16] equated the adhesive force of a bubble to its buoyancy at the break-off moment, and introducing experimental coefficients, arrived at the following semi-empirical formula for the average radius of a departing bubble

$$R_0 = 0{\cdot}0104\, \theta\, [\sigma/g\, (\rho_L - \rho_v)]^{\frac{1}{2}} \tag{3-3}$$

In this formula the bubble is considered to have a spherical shape at its departure. Verification, by M. Jakob and co-workers, of the formula has shown that it is in accordance with a statistical distribution of the then available experimental data [20].

Bošnjaković in his book on *Thermodynamics*, and shortly after him M. Jakob, proposed a simple model of nucleate boiling by supposing that the latent heat penetrating by vaporization into the bubble is equal to the surface heat transfer at the interface. This is expressed by equation (3-2).

According to these authors "generally a vapour bubble will be surrounded by a very thin boundary layer of liquid across which the temperature decreases from the higher value existing in the superheated liquid to the saturation value."

This means that in the opinion of these authors the rising bubble does not carry with it its own boundary layer, but simply that the slightly superheated bulk liquid feeds the bubble with heat and vapour. The boundary layer of the bubble would therefore appear and disappear almost instantly, corresponding to the momentary position of the bubble.

These assumptions lead to the equation

$$\mathrm{d}Q = L\,\rho_v\,\mathrm{d}V = hA\,(t_b - t_s)\,\mathrm{d}\tau \quad (3\text{-}4)$$

where t_b is the bulk temperature and t_s the saturation temperature of the vapour.

For a spherical bubble equation (3-4) becomes

$$\dot{R} = h\,(t_b - t_s)/L\,\rho_v. \quad (3\text{-}5)$$

The same model was used by Fritz and Ende [27] in their derivation of the formula

$$R = 2\,\frac{(t_w - t_s)\,c_L\,\rho_L}{L\rho_v}\,(a\tau/\pi)^{\frac{1}{2}}. \quad (3\text{-}6)$$

Similar assumptions lead to

$$\dot{R} = \frac{Ck\,(t_w - t_s)}{L\,\rho_v}\,(\pi a\tau)^{-\frac{1}{2}} \quad (3\text{-}7)$$

with $C = \sqrt{3}$ according to Plesset and Zwick [17]

and $C = \pi/2$ according to Forster and Zuber [18].

The above formulae are based on the assumption that the vapour–liquid interface of a bubble may be regarded as a plane surface.

The curvature of the interface was first taken into account by Carslaw and Jaeger [19] who found for a spherical hole situated in an infinite, uniformly superheated bulk of liquid, the following temperature gradient

$$\left(\frac{\partial t}{\partial r}\right)_{r=R} = (t_b - t_s)\,[(\pi a\tau)^{-\frac{1}{2}} + 1/R], \quad (3\text{-}8)$$

an expression which enabled Forster [12] to improve the formula of Fritz (3-6) by introducing the correction factor

$$C = \tfrac{1}{2}\,[1 + (1 + 2\,\pi/Ja)^{\frac{1}{2}}] > 1, \quad (3\text{-}8')$$

where the dimensionless "modulus of Jakob"

$$Ja = C_L\,\rho_L\,(t_w - t_s)/L\,\rho_v \quad (3\text{-}9)$$

represents a characteristic group appearing in most of the formulae regarding the evolution of the bubbles. Formula (3-8′), already represents remarkable progress, but it does not take into account the motion of the bubble and that of the surrounding liquid. For the moderate velocities frequently encountered, these influences are negligible and the formula may be used as a satisfactory approximation. This is no longer the case at higher velocities, since then the ascending and descending currents associated with the motion of the bubble must strongly influence the growth and the motion of the bubble.

Returning to equation (3-2) we shall assume that the bubbles carry with them thin layers of superheated liquid, broken away at their departure from the layer of the heating surface. We shall further assume that the temperature field of such a boundary layer depends upon the following quantities: the radial co-ordinate $(r-R)$ and the corresponding reference length $(R'-R)$, for the boundary layer; the characteristic temperature difference $\Delta t = (t_w - t_s)$ and the reference length R_0, for the departing bubble; the physical parameters of the medium c_L, L, a, ρ_L, ρ_v; the time τ.

Dimensional analysis leads to formal expressions of the following kind for the temperature field within the boundary layer of the bubble:

$$[t] = \text{const.}\,\Delta t \times Ja^{m} \times (a\tau/R_0^2)^{m'} \times [(r - R)/(R' - R)]^{m''} \quad (3\text{-}10)$$

The solutions of (3-2) formed on the basis of (3-10) must satisfy the following boundary conditions

$$\left.\begin{array}{ll} t_{r=R} = t_s; & t_{r=R'} = t_b; \\ (\partial t/\partial r)_{r=R} \neq 0; & (\partial t/\partial r)_{r=R'} = 0. \end{array}\right\} \quad (3\text{-}11)$$

The last condition $(\partial t/\partial r)_{r=R'} = 0$ expresses the fact that the transition from the bulk to the boundary layer of the bubble is smooth, without temperature gradient, as shown in Fig. 5.

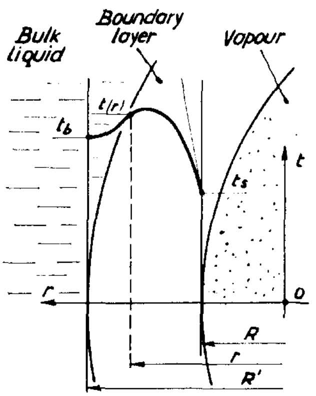

FIG. 5

We will now build up a suitable and sufficiently general expression for the temperature field across the boundary layer of the bubble, which fits the conditions (3-11). For the sake of simplicity we shall introduce the dimensionless spatial variable

$$x = (r - R)/(R' - R), \quad 0 \leqslant x \leqslant 1, \quad (3\text{-}12)$$

observing that

$$(\partial/\partial r) = \frac{1}{R' - R} \cdot (\partial/\partial x) = \frac{1}{\kappa R} \cdot (\partial/\partial x).$$

The linear term

$$t_1 = x \cdot t_b + (1 - x) \cdot t_s$$

gives $t_1 = t_s$ when $x = 0$, and $t_1 = t_b$ when $x = 1$, but the gradient is

$$\partial t_1/\partial r = (t_b - t_s)/\kappa R \neq 0$$

In order to obtain the desired zero gradient, without affecting the values of the temperature on the two interfaces of the layer, we introduce the following additional term

$$t_2 = (t_b - t_s)\, x\, (1 - x)\, (2x - 1)$$

and finally a term giving the profile of the temperature curve

$$t_3 = \Delta t \cdot x(1 - x)^{1+\alpha} \cdot Ja^m \sum_n f_n(x) \cdot g_n(a\tau/R_0^2)$$

where the functions $f_n(x)$ and $g_n(a\tau/R_0^2)$ are continuous within the intervals of variation of their arguments, but so far undetermined. The exponent $\alpha > 0$, has been introduced in order to maintain a nul temperature gradient on the bulk side of the boundary layer.

The desired expression for the temperature profile is therefore

$$t = \sum_{i=1}^{3} t_i = x\, t_b + (1 - x)\, t_s + (t_b - t_s)\, x\, (1 - x)\, (2\, x - 1) + \Delta t\, x\, (1 - x)^{1+\alpha}\, Ja^m \times \sum_n f_n\, (x)\, g_n\, (a\tau/R_0^2). \quad (3\text{-}13)$$

The temperature gradient at the vapour-layer interface is

$$(\partial t/\partial r)_{r=R} = \frac{\Delta t}{\kappa R}\, Ja^m \sum_n f_n(0)\, g_n(a\tau/R_0^2). \quad (3\text{-}14)$$

Making the substitution

$$F\,(a\tau/R_0^2) = \sum_n f_n(0)\, g_n(a\tau/R_0^2), \quad (3\text{-}15)$$

relation (3-14) becomes

$$(\partial t/\partial r)_{r=R} = \frac{\Delta t}{\kappa R}\, Ja^m\, F(a\tau/R_0^2). \quad (3\text{-}16)$$

Introducing this expression in equation (3-2) we obtain

$$R\, \mathrm{d}R = \frac{k\, \Delta t}{\kappa\, L\, \rho_v}\, Ja^m\, F(a\tau/R_0^2)\, \mathrm{d}\tau. \quad (3\text{-}17)$$

Integration of this equation gives

$$R^2 = \frac{2}{\kappa}\, Ja^{m+1} \cdot R_0^2 \int_0^{z = a\tau/R_0^2} F(z) \cdot \mathrm{d}z, \quad (3\text{-}18)$$

where the time origin was made coincident with the bubble origin, i.e.

$$R = 0 \quad \text{at} \quad \tau = 0. \qquad (3\text{-}19)$$

The most simple case is obtained by putting

$$F(z) = C = \text{constant}, \qquad (3\text{-}20)$$

which leads to the very important result

$$R^2 = \frac{2C}{\kappa} Ja^{m+1} a\tau \qquad (3\text{-}21)$$

which may be compared with the formula (3-6) of Fritz and Ende, written in the form

$$R = 2\, Ja\, (a\tau/\pi)^{\frac{1}{2}}. \qquad (3\text{-}21')$$

Comparison shows that with

$$m = 1 \quad \text{and} \quad C = 2\kappa/\pi$$

the two formulae become identical.

More general expressions for the radius R of the bubble may be obtained by substituting

$$F(z) = Bz^p \qquad (3\text{-}22)$$

into (3-18); this leads to

$$(R/R_0)^2 = \frac{2B}{\kappa(p+1)} Ja^{m+1} (a\tau/R_0^2)^{p+1}. \quad (3\text{-}23)$$

Putting

$$p + 1 = 2/n, \quad m + 1 = 2/q \qquad (3\text{-}24)$$

we obtain

$$R/R_0 = C \,.\, \kappa^{-\frac{1}{2}} Ja^{1/q}(a\tau/R_0^2)^{1/n} \qquad (3\text{-}25)$$

where

$$C = (n\, B)^{\frac{1}{2}}.$$

This is a wide-range expression for the radius–time relationship, particular cases of which have already been obtained by different authors.

So, for example, Forster [12], assuming an exponential temperature distribution across the boundary layer of the heating surface, i.e.

$$t - t_s = (t_w - t_s) \exp(-x/H),$$

where H is a reference length, obtained the following formula for bubbles which have grown larger than the thickness of the boundary layer of the heating surface

$$R \sim 2\, (H\, Ja)^{\frac{1}{2}}\, (a\tau/\pi)^{1/4}.$$

This expression may be deduced from (3-25) by putting

$$q = 2 \quad \text{and} \quad n = 4.$$

Likewise, the formula of Fritz and Ende (3-21′) results from substituting into (3-25)

$$q = 1 \quad \text{and} \quad n = 2.$$

Griffith [21] found that n varies between 2 and 4·5 and Staniszewsky [22] that it varies between 1 and 3, the smaller values being erferred to the beginning, the larger to the end of the growth of the bubble.

The relation (3-25) may be written in the alternative form

$$R/R_0 = (\kappa_0/\kappa)^{\frac{1}{2}}\, (\tau/\tau_0)^{1/n} \qquad (3\text{-}26)$$

if within the interval τ_0 to τ the numbers q and n do not significantly change.

It seems necessary, nevertheless, to comment that the macroscopic considerations used in order to obtain formula (3-25) do not permit the values of C, q and n to be determined by calculus.

4. RISING VELOCITY OF VAPOUR BUBBLES

The bubble, together with its apparent mass, will be considered as a whole and called a "particle". Owing to the small size of such a particle, it will be treated as a mass-point.

The virtual mass of a bubble is

$$M_v = M + M_a = \tfrac{4}{3}\, \pi\, R^3\, (\rho_v + \mu\, \rho_L);$$

its buoyancy is

$$F = g\, (\rho_L - \rho_v)\, V = \tfrac{4}{3}\, \pi\, R^3\, g\,.\, (\rho_L - \rho_v)$$

where g is the acceleration due to gravity; the frictional drag on the bubble is

$$W = \zeta\, A'_{tr}\, \rho_L\, v^2/2$$

where

A'_{tr} = area of the transverse section of a particle;

ζ = coefficient of friction for $Re = 2R'v/\nu$.

In order to simplify the calculus, the particle will be assumed to be of spherical shape. Since $Re \approx 2000$, resistance is quadratic and $\zeta \approx 0{\cdot}55$ [23].

The momentum equation of such a particle is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(M_v v) = \frac{\mathrm{d}}{\mathrm{d}\tau}[(M + M_a) v] = F - W \quad (4\text{-}1)$$

Introducing the values of M_v, F and W as well as the new constants

$$\left.\begin{aligned} g_1 &= g(\rho_L - \rho_v)/(\rho_v + \mu\rho_L) \\ \zeta_1 &= \tfrac{3}{8}\,\zeta(1+\mu)^{2/3}\,\rho_L/(\rho_v + \mu\rho_L) \end{aligned}\right\} \quad (4\text{-}2)$$

equation (4-1) takes the form

$$\dot{v} + 3\,(\dot{R}/R)v = g_1 - \zeta_1\, v^2/R \quad (4\text{-}3)$$

This Ricatti equation has to be integrated under the initial conditions

$$\tau = 0, \quad v = 0. \quad (4\text{-}4)$$

In order to integrate this equation it is necessary to know either R as a function of the time τ or the velocity v as a function of R and τ.

Two different phases of the motion of a bubble must be distinguished, namely (a) the bubble still adhering to the plate, before its break off; and (b) the free ascending bubble.

We begin with the second phase, that of the free rising bubble. We introduce into equation (4-3) the expression (3-26) for the radius R, which we write in the simpler form

$$R = b\,\tau^{1/n} \text{ with } b = (\kappa_0/\kappa)^{\frac{1}{2}}\,\tau_0^{-1/n}. \quad (4\text{-}5)$$

Hence

$$\dot{v} + \frac{3}{n}\,\tau^{-1}\,v = g_1 - \frac{\zeta_1}{b}\,\tau^{-1/n}\,v^2. \quad (4\text{-}6)$$

The integration of this equation is shown in the appendix. The solution, corresponding to condition (4-4) is

$$v = (g_1\, b/\zeta_1)^{\frac{1}{2}}\,\tau^{1/2n} \,.\, I_{p+1}(qx)/I_p(qx) \quad (4\text{-}7)$$

where

$$p = (4-n)/(2n-1), \quad x = \tau^{(2n-1)/2n},$$

$$q = \frac{2n}{2n-1}\,(\zeta_1 g_1/b)^{\frac{1}{2}}.$$

The asymptotic value of (4-7) for $\tau \to \infty$ is, as shown in the appendix,

$$v = (g_1\, b/\zeta_1)^{\frac{1}{2}} \,.\, \tau^{1/2n}. \quad (4\text{-}8)$$

It is possible to reach solution (4-8) directly, by considering equation (4-3) and examining the relative importance of its different terms.

Indeed the terms on the left-hand side of (4-3) concerning the inertial effects, are negligible in comparison with the buoyancy or the resistance terms. Neglecting the left-hand side of equation (4-3) this differential equation reduces to the algebraic equation

$$g_1 - \zeta_1\, v^2/R = 0 \quad (4\text{-}9)$$

the solution of which is

$$v = (g_1\, R/\zeta_1)^{\frac{1}{2}} \quad (4\text{-}10)$$

and introducing expression (4-5) for the radius R, the solution becomes

$$v = (g_1\, b/\zeta_1)^{\frac{1}{2}}\,\tau^{1/2n} \quad (4\text{-}10')$$

which is identical to (4-8).

As previously shown, the number n takes the value 1 to 4, depending on the stage of the motion of the bubble.

We now turn our attention to the first phase of the evolution of a bubble, i.e. the period during which the growing bubble is still attached to the plate. As shown in Fig. 3, during this stage we may put

$$v \approx \dot{R}. \quad (4\text{-}11)$$

Indeed, since the bubble is of an almost spherical shape its centre of gravity is at a height h above the plate

$$h \approx R$$

so that the velocity of the centre of gravity is

$$v = \dot{h} \approx \dot{R}$$

as assumed.

Introducing equation (4-11) into (4-3) we obtain

$$R\,.\,\ddot{R} + (3 + \zeta_1)\,\dot{R}^2 - g_1\, R = 0. \quad (4\text{-}12)$$

The initial condition remains the same as before

$$\tau = 0, \quad R = 0 \quad (4\text{-}4)$$

so that we may try a solution of the form

$$R = c\,.\,\tau^m \quad (4\text{-}13)$$

which gives

$$m = 2, \quad c = g_1/2\,(2\,\zeta_1 + 7)$$

and subsequently

$$\left.\begin{aligned} R &= g_1\,\tau^2/2\,(2\,\zeta_1 + 7), \\ v &= g_1\,\tau/(2\,\zeta_1 + 7). \end{aligned}\right\} \quad (4\text{-}14)$$

From these two equations, by eliminating the time variable we obtain

$$v = [g_1 R/(\zeta_1 + 3{\cdot}5)]^{\frac{1}{2}} \quad (4\text{-}15)$$

an expression similar to (4-8) and (4-10).

By division we obtain the very simple relation

$$v = 2R/\tau. \quad (4\text{-}16)$$

The equations (4-14) show a quadratic time dependence of the radius and a linear time dependence of the velocity.

Since in general

$$\zeta_1 \ll 3{\cdot}5$$

comparison of equations (4-10) and (4-15) shows a sudden jump in the velocity of the bubble from shortly before the departure to shortly after it.

This seems to be in contradiction to the observations of Max Jakob and co-workers, who report that between the velocity of the attached bubble and that of the departing one, there is no appreciable difference [15].

In the next section it will be shown that there is no contradiction between the statement of M. Jakob and the equations (4-14) to (4-16), since the motion of the attached bubble must in turn be divided into two different phases.

Returning for an instant to equation (4-10), which was deduced for very large values of the time parameter, or for negligible inertial effects, we will assume that it remains valid for the departure conditions of a bubble. For this moment, equation (4-10) becomes

$$v_0 = (g_1 R_0/\zeta_1)^{\frac{1}{2}} \quad (4\text{-}17)$$

and introducing the value of R_0 given by Fritz, i.e.

$$R_0 = 0{\cdot}0104\, \theta\, [\sigma/g\,(\rho_L - \rho_v)]^{\frac{1}{2}} \quad (3\text{-}3)$$

we obtain for the velocity of the departing bubble

$$v_0 = 0{\cdot}102\, (g_1\, \theta/\zeta_1)^{\frac{1}{2}}\, [\sigma/g\,(\rho_L - \rho_v)]^{1/4}. \quad (4\text{-}18)$$

From equations (4-2) we have

$$g_1/\zeta_1 = \tfrac{8}{3}\, (g/\zeta)\, \frac{\rho_L - \rho_v}{\rho_L}\, (1 + \mu)^{-2/3}$$

so that

$$v_0 = 0{\cdot}167\, (\theta/\zeta)^{\frac{1}{2}}\, (1 + \mu)^{-1/3}\, [\sigma g\,(\rho_L - \rho_v)/\rho_L^2]^{1/4}, \quad (4\text{-}19)$$

and with

$$\mu \approx 1{\cdot}5 \quad \text{and} \quad \zeta \approx 0{\cdot}55$$

we obtain

$$v_0 = 0{\cdot}166\, \theta^{\frac{1}{2}}\, [\sigma g\,(\rho_L - \rho_v)/\rho_L^2]^{1/4}. \quad (4\text{-}20)$$

It will be shown in the next section that, in spite of the extension of the domain of validity, formula (4-20) is in satisfactory agreement with experimental data.

5. ANALYSIS OF RESULTS AND CONCLUSIONS

The results attained in the previous sections were based on the following main assumptions:

ascending bubbles carry around them boundary layers of superheated liquid, broken away at their departure, from the boundary layer of the heating surface;

the mass of such a layer is, at any moment, of the same order of magnitude as the corresponding apparent mass of the bubble.

In order to ascertain how closely these assumptions may approach reality and lead to useful results, some of the results obtained will be compared with corresponding experimental data.

By extending equation (4-10) far beyond its range of validity, we obtained equation (4-20), i.e.

$$v_0 = 0{\cdot}166\, \theta^{\frac{1}{2}}\, [\sigma g\,(\rho_L - \rho_v)/\rho_L^2]^{1/4}.$$

If values for boiling water at 1 atm pressure are introduced into this relation, i.e.

$$\theta = 50^\circ$$

then the formula becomes

$$v_0 = 1{\cdot}17\ .\ [\sigma g\,(\rho_L - \rho_v)/\rho_L^2]^{1/4} \quad (5\text{-}1)$$

which is the formula of Peebles and Garber [25], obtained experimentally and with the slightly different numerical coefficient of 1·18.

In his book on *Heat Transfer* [15] Max Jakob reproduces (p. 633, Fig. 29-14) some experimental data regarding the height h attained at different moments during an interval of about 0·20 s by vapour bubbles in boiling water at 1 atm pressure. The figure indicates also certain statistical mean velocities v of the rising bubbles. From this figure we have extracted the data which are shown in Table 1. We shall

Table 1

Time τ (s)	Height, h (cm)	Velocity, v (cm/s)	Observations
0·05	0·75	17	break-off
0·10	1·67	—	—
0·125	2·20	23	—
0·20	4·40	33	—

compare these data with values derived from the formulae developed above.

The break-off velocity of the bubble may be obtained either from equation (4-17) or from (5-1) which give the same value. Introducing into (4-17) $\tau_0 = 0{\cdot}05$ as given by M. Jakob, $\bar{\mu} = 1{\cdot}50$ and $R_0 = 0{\cdot}125$ cm (as resulting from formula (3-3) of Fritz) we obtain with

$$g_{10} = 656 \quad \text{and} \quad \zeta_{10} = 0{\cdot}25$$

the value

$$v_0 = (g_{1\cdot 0}\, R_0/\zeta_{3\cdot 0})^{\frac{1}{2}} = (656 \times 0{\cdot}125/0{\cdot}25)^{\frac{1}{2}} = 18 \text{ cm/s}$$

This velocity differs by less than 6 per cent from the value given by Jakob in the above-mentioned figure. It will be used as a reference value in the further calculations.

Since the interval of 0·20 s is not too large, it is permissible to assume that during this interval exponents $1/q$ and $1/n$ in equation (3-25) remain practically unchanged. By combining relations (4-10), (4-17) and (3-26) we deduce therefore

$$v = v_0\,(\kappa_0/\kappa)^{1/4} \,.\, \left[\frac{g_1/\zeta_1}{g_{10}/\zeta_{10}}\right]^{\frac{1}{2}} .\, (\tau/\tau_0)^{1/2n} \qquad (5\text{-}2)$$

and put for the whole interval

$$n = 2 \qquad (5\text{-}3)$$

As for the coefficient of apparent mass $\bar{\mu}$, for the interval $\tau = 0{\cdot}05$ to $0{\cdot}15$ s, we will put

$$\bar{\mu} = 1{\cdot}5 \qquad (1\text{-}7)$$

for $\tau = 0{\cdot}20$ s, since then the velocity of the rising bubble has attained approx. 0·30 m/s and the distance of the bubble from the plate is about 5 cm, we may put

$$\bar{\mu} = \mu_t = 1{\cdot}34 \qquad (1\text{-}8)$$

The height attained by the bubble will be approximated by putting

$$h \approx \int_0^\tau v\, d\tau = \int_0^\tau c\, \tau^{1/2n}\, d\tau = \frac{2n}{2n-1}\, c\, \tau^{1+1/2n} = \frac{2n}{2n-1}\, v\, \tau \qquad (5\text{-}4)$$

Putting $n = 2$ and $v_0 = 18$ cm/s leads to the data of Table 2

It is easily seen that in spite of the many simplifying assumptions, formulae (4-10) and (3-25) lead to satisfactory agreement with experimental data, since differences are less than 6 per cent for the velocities and less than 13 per cent for the attained heights.

The above considerations were all for free ascending bubbles.

We turn now our attention to attached bubbles, just before departure. Equation ($4\text{-}14_2$), i.e.

$$v = g_1\, \tau/(2\, \zeta_1 + 7),$$

gives for the break-off moment

$$v_0 = 4{\cdot}4 \text{ cm/s}, \qquad (5\text{-}5)$$

whereas Max Jakob in reference [15], as well as other authors, states that towards the break-off

Table 2

Time τ (s)	$\bar{\mu}$	κ	g_1/ζ_1 (cm/s²)	velocity v_{calc} (cm/s)	velocity v_{meas} (cm/s)	height h_{calc} (cm)	height h_{meas} (cm)
0·05	1·5	0·36	2624	18	17	0·72	0·75
0·10	1·5	0·36	2624	21·3	—	1·70	1·67
0·125	1·5	0·36	2624	22·6	23	2·26	2·20
0·20	1·34	0·32	3730	31·2	33	5·00	4·40

moment the velocity of attached bubbles is

$$v_0 = 17 \text{ cm/s} \qquad (5\text{-}5')$$

i.e. almost the same as after the break-off. This discrepancy is nevertheless easily explained as follows.

From relation (4-16) we deduce

$$R = v\tau/2 \qquad (5\text{-}6)$$

so that for the break-off moment with the experimental data

$$\tau_0 = 0{\cdot}05 \text{ s}$$

as given by Jakob, and velocity as resulting from (5-5)

$$v_0 = 4{\cdot}4 \text{ cm/s},$$

we obtain

$$R_0 = 0{\cdot}11 \text{ cm}; \qquad (5\text{-}7)$$

whereas the semi-empiric formula (3-3) of Fritz gives

$$R_0 = 0{\cdot}125 \text{ cm},$$

a very close value for a statistical quantity.

Now returning to the above-mentioned discrepancy, we shall show that as long as the attached bubble keeps a spherical shape, formulae (4-14) remain fully valid.

During an interval 0 to τ, ($\tau \leqslant \tau_0$), the centre of a spherically shaped, attached bubble attains a height

$$h \approx R, \qquad (5\text{-}8)$$

so that its mean rising velocity is

$$v_m = h/\tau \approx R/\tau. \qquad (5\text{-}9)$$

From equations (4-14) it follows that

$$v_m = \frac{1}{\tau}\int_0^\tau v \,\mathrm{d}\tau = g_1\,\tau/2\,(2\,\zeta_1 + 7) = R/\tau$$

i.e. the same as above.

Formulae (4-14) are therefore correct and valid, but *only for spherical bubbles, directly attached to the heating plate.*

But as observed by Max Jakob and others, and shown for example in reference [15] (p. 629, Fig. 29-12), the bubbles, *shortly before leaving the heating surface, generally form a stem-like prolongation* by which they remain attached to the plate, whilst their spherical part begins to rise, permitted by the great extensibility of the stem.

The adhesive force, retaining the bubble to the surface, is

$$F' \approx 2\,\pi\,r\,\sigma \qquad (5\text{-}10)$$

where r is the radius of the transverse section of the stem, near the hot plate (Fig. 6).

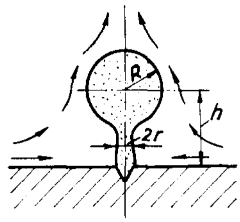

FIG. 6

The motion of the bubble remains governed by an equation similar to (4-1) but with an extra term allowing for the above adhesive force, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\,(M_v\,.\,v) = F - W - F'. \qquad (5\text{-}11)$$

As previously shown, the inertial forces, on the left-hand side of this equation, are negligible, compared with the buoyancy and frictional forces, so that they may be omitted. Equation (5-11) reduces to

$$F - W - F' = 0 \qquad (5\text{-}12)$$

But it is easy to show that the adhesive force F' is also negligible compared with the buoyancy or the friction, since the stems of the bubbles are, comparatively, very thin, i.e. $r \ll R$.

We may therefore write without much error

$$F - W = 0.$$

But this is equation (4-9) and leads to solution (4-10), i.e. to values of the same order of magnitude as those observed by M. Jakob and coworkers. The observations of these authors are therefore confirmed also by calculus.

The above analysis of the behaviour of attached bubbles has the advantage of permitting an important conclusion, namely that the motion of attached bubbles before their departure, is a step-by-step process and that the first step, regarding the growth of spherical bubbles, directly attached to the heating surface is governed by the two equations (4-14), and the second step, concerning stalked bubbles, depends approximately on equation (4-10).

The satisfactory agreement reached when comparing some experimental data with the corresponding data resulting from the formulae deduced in this paper, may be considered as an encouragement to take into account the apparent masses of the rising bubbles, and this although the inertial effects are rather moderate, at least at the beginning of the rise. Examination of more extensive data than those available to author, would certainly lead to a deeper insight into the phenomenon of bubble evolution and would permit improvement of the attained results.

From this point of view some comments must be made with respect to the character of some of the parameters entering into the established formulae.

First the "radius" R of the bubble, which is in fact the quantity

$$R = (3V/4\pi)^{1/3}$$

which may be considered as representative of the volume but in no case of the shape of the bubble.

In a certain measure the coefficient of apparnet mass $\bar{\mu}$ may act as a corrective in this direction, but its appreciation is difficult since no experimental data are so far available and the comparison with a flattened ellipsoid is valid only for the begining of the rise. For mushroom-like bubbles this approach cannot be maintained.

The exponents $1/q$ and $1/n$ as well as the numerical coefficient C in formula (3-25) cannot be determined from phenomenological considerations. The same is true for the frequency of the bubble production at a given centre or for the distribution of active centres over the heating surface.

In conclusion, though the developed formulae may be regarded as being in satisfactory agreement with experimental data, more precision cannot be reached without having recourse to molecular theories and to experiments devised to obtain a law concerning the variation of the shape of the bubbles during their rising motion.

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APPENDIX

Integration of Differential Equation (4-6)

The general equation of Ricatti, i.e.

$$\dot{v} = P(\tau) + v\,Q(\tau) + v^2\,R(\tau)$$

reduces to a linear equation of the second order if one puts

$$v = -\frac{1}{R(\tau)}\,\dot{y}/y$$

resulting in

$$\ddot{y} - (\dot{R}/R + Q)\,\dot{y} + P\,Ry = 0.$$

Equation (4-6) is

$$\dot{v} + \frac{3}{n}\,\tau^{-1}\,v = g_1 - \frac{\zeta_1}{b}\,\tau^{-1/n}\,v^2 \qquad \text{(A-1)}$$

thus by putting

$$v = \frac{b}{\zeta_1}\,\tau^{1/n}\,\dot{y}/y \qquad \text{(A-2)}$$

and substituting into (A-1) the result is

$$\ddot{y} + \frac{4}{n}\,\tau^{-1}\,\dot{y} - \frac{g_1\,\zeta_1}{b}\,\tau^{-1/n}\,y = 0. \qquad \text{(A-3)}$$

This is a generalized Bessel equation of the form

$$\ddot{y} + \frac{1-2a}{\tau}\,\dot{y} + \left[(\beta\,.\,\gamma\,.\,\tau^{\gamma-1})^2 + \frac{a^2 - p^2\,\gamma^2}{\tau^2}\right].\,y = 0$$

the general solution of which is [26]

$$y = \tau^{a}\,Z_p\,(\beta\tau^{\gamma}),$$

where Z is a Bessel function satisfying the integration conditions.

Since

$$1 - 2a = 4/n;\quad 2(\gamma - 1) = -1/n;$$
$$(\beta\gamma)^2 = -\zeta_1\,g_1/b;\quad a^2 - p^2\,\gamma^2 = 0$$

the solution of (A-3) is

$$y = \tau^{n-4/2-n}\,.\,Z_p\left[\frac{2n}{2n-1}\,.\,i\,.\,(\zeta_1\,g_1/b)^{\frac{1}{2}}\,.\,\tau^{2n-1/2n}\right]. \qquad \text{(A-4)}$$

Making the following substitutions

$$\left.\begin{aligned} x &= \tau^{(2n-1)/2n} \\ p &= (4-n)/(2n-1) \\ q &= \frac{2n-1}{2n}\,(g_1\,\zeta_1/b)^{\frac{1}{2}} \end{aligned}\right\} \qquad \text{(A-5)}$$

solution (A-4) becomes

$$y = x^{-p}\,.\,Z_p\,(iqx) \qquad \text{(A-6)}$$

and since [26]

$$\frac{d}{dx}\,[x^{-p}\,.\,Z_p\,(iqx)] = -x^{-p}\,.\,Z_{p+1}\,(iqx)\,.\,iq \qquad \text{(A-7)}$$

it follows that

$$\dot{y}/y = -iq\dot{x}\,Z_{p+1}\,(iqx)/Z_p\,(iqx). \qquad \text{(A-8)}$$

In order to ensure a finite value of the velocity v for $\tau = 0$ (i.e. $x = 0$) we choose for Z_p the Bessel function of the first kind, i.e.

$$J_p\,(iqx) = i^p\,I_p\,(qx)$$

so that finally

$$v = (g_1\,\gamma/\zeta_3)^{\frac{1}{2}}\,.\,\tau^{1/2n}\,.\,I_{p+1}\,(qx)/I_p\,(qx). \qquad \text{(A-9)}$$

Since for $\tau \to \infty$

$$\lim_{\tau\to\infty} I_p\,(qx) = e^{qx}/(2\pi\,qx)^{\frac{1}{2}}$$

which is independent of the value of the index p it results that

$$\lim_{\tau\to\infty} v = (g_1\,b/\zeta_1)^{\frac{1}{2}}\,.\,\tau^{1/2n} \qquad \text{(A-10)}$$

which is expression (4-8) or (4-10′).

Résumé—Les hypothèses suivantes ont été faites dans le cas de l'ébullition par nucléation en réservoir dans le régime de bulles de vapeur individuelles: (a) les bulles montantes emportent evac elles de fines couches limites de liquide surchauffé, brisé au moment de leur départ, à partir de la couche limite de la surface chauffante; (b) les perturbations produites dans le sein du fluide sont les mêmes que si les bulles étaient des corps solides de même taille et de même forme.

En employant la théorie des masses virtuelles et les hypothèses précédentes, on a exposé des relations pour la vitesse de croissance et la vitesse d'ascension des bulles, qu'on a trouvé être en accord satisfaisant avec les données expérimentales disponibles.

Zusammenfassung—Die folgenden Annahmen sind für den Fall des Blasensiedens bei freier Konvektion im Bereich der einzeln auftretenden Dampfblasen getroffen: (a) aufsteigende Blasen tragen eine dünne Grenzschicht überhitzter Flüssigkeit mit sich, die sie beim Ablösen aus der Grenzschicht an der Heizfläche mit herausreissen; (b) die Störungen, die von den Blasen in der umgebenden Flüssigkeit verursacht werden, sind die gleichen, wie wenn die Blasen feste Körper von gleicher Form und Grösse wären.

Unter Verwendung virtueller Massen und den obigen Annahmen werden Beziehungen für das Anwachsen und die Steiggeschwindigkeit der Blasen entwickelt, die sich als zufriedenstellend übereinstimmend mit verfügbaren Versuchswerten erweisen.

Аннотация—Сделаны следующие допущения для случая пузырькового кипения в большом объеме в режиме отдельных пузырьков пара : (а) поднимающиеся пузырьки выносятся вместе с тонким пограничным слоем перегретой жидкости, отрывающимся при отделении пузырьков от поверхности нагрева; (*б*) возмущения внутри жидкости, вызванные пузырьками, совершенно аналогичны воздействиям твердых частиц такой же формы и размера.

С помощью теории эффективных масс и вышеуказанных допущений, получены уравнения для скорости роста и подъема пузырьков, дающие удовлетворительное согласование с известными экспериментальными данными.